The solution to a conjecture of Tits on the subgroup generated by the squares of the generators of an Artin group

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Abstract

Let A be an Artin group with standard generating set $\{\sigma_s : s \in S\}$. Tits conjectured that the only relations in A amongst the squares of the generators are consequences of the obvious ones, namely that σ_s^2 and σ_t^2 commute whenever σ_s and σ_t commute, for $s, t \in S$. In this paper we prove Tits' conjecture for all Artin groups. In fact, given a number $m_s \geq 2$ for each $s \in S$, we show that the elements $\{T_s = \sigma_s^{m_s} : s \in S\}$ generate a subgroup that has a finite presentation in which the only defining relations are that T_s and T_t commute if σ_s and σ_t commute.

1 Introduction

Let S be a finite set. A **Coxeter matrix** over S is a matrix $M = (m_{s,t})_{s,t \in S}$ indexed by the elements of S and such that:

(a) $m_{s,s} = 1$ for all $s \in S$, and
(b) $m_{s,t} = m_{t,s} \in \{m \in \mathbb{N} : m \geq 2\} \cup \{\infty\}$ for all $s \neq t \in S$.

A Coxeter matrix is represented by its **Coxeter graph** $\Gamma = \Gamma(M)$, which is defined as follows:

(a) S is the set of vertices of Γ;
(b) two vertices s and t are joined by an edge if $s \neq t$ and $m_{s,t} \neq 2$;
(c) the edge joining s and t is labelled $m_{s,t}$ if $m_{s,t} \geq 4$ or $m_{s,t} = \infty$.

Take an abstract set $\Sigma = \{\sigma_s : s \in S\}$ in bijection with S. For $s, t \in S$ and $m \in \mathbb{N}$ define the word

$$\mathrm{prod}(\sigma_s, \sigma_t; m) = \begin{cases} (\sigma_s \sigma_t)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (\sigma_s \sigma_t)^{\frac{m-1}{2}} \sigma_s & \text{if } m \text{ is odd.} \end{cases}$$

The **Artin system** associated to $\Gamma = \Gamma(M)$ is the pair $(A(\Gamma), \Sigma)$ where $A(\Gamma)$ is the group defined by the presentation

$$\langle \Sigma \mid \mathrm{prod}(\sigma_s, \sigma_t; m_{s,t}) = \mathrm{prod}(\sigma_t, \sigma_s; m_{s,t}), \text{ for all } s, t \in S \text{ with } m_{s,t} \neq \infty \rangle.$$

This paper is concerned with subgroups of the Artin group $A = A(\Gamma)$ which are generated by powers of the generators. In [1], Appel and Schupp showed that, in the case of so-called "extra-large type" Artin groups (i.e: where $m_{s,t} \notin \{2, 3\}$ for $s, t \in S$), the set of squares of generators $\mathcal{Q} = \{Q_s = \sigma_s^2 : s \in S\}$ freely generates a free subgroup of A. They then stated the conjecture, communicated to them by Tits, that for an arbitrary Artin group the obvious relations, $Q_s Q_t = Q_t Q_s$ if $m_{s,t} = 2$, might be sufficient to give a presentation of the subgroup generated by $\mathcal{Q}$. This conjecture was apparently motivated by recalling a much earlier result of Tits [15, Théorème 2.5] which says, essentially, that the abelianization of the pure Artin group maps the squares of the generators of A to a set of independent generators of a free abelian group, showing that the subgroup in question lies somewhere between the group $H = \langle \mathcal{Q} \mid Q_s Q_t = Q_t Q_s \text{ if } m_{s,t} = 2 \rangle$ and the abelianization of H.

In this paper we prove the following theorem:

Theorem 1. *Let Γ be a Coxeter graph over S, and suppose that one is given an integer $m_s \geq 2$ for each $s \in S$. Then the set of elements $\mathcal{T} = \{T_s = \sigma_s^{m_s} : s \in S\}$ generates a subgroup of $A(\Gamma)$ with presentation*

$$\langle \mathcal{T} \mid T_s T_t = T_t T_s \text{ if } m_{s,t} = 2 \rangle.$$

Corollary 2. *The Tits Conjecture holds for every Artin system. Namely, given an arbitrary Coxeter graph Γ, the set of squares of the Artin generators, $\mathcal{Q} = \{Q_s = \sigma_s^2 : s \in S\}$, generates a subgroup of $A(\Gamma)$ with presentation*

$$\langle \mathcal{Q} \mid Q_s Q_t = Q_t Q_s \text{ if } m_{s,t} = 2 \rangle.$$

After the paper of Appel and Schupp, the Tits Conjecture was proven in various special cases: by Pride [14] for the "triangle-free" groups (i.e: those in which every three generators include a pair s, t for which $m_{s,t} = \infty$); by Droms, Lewin and Servatius [8] for braid groups on fewer than 6 strings; by Collins [6] for all braid groups; by Humphries [13] in some restricted cases where $m_{s,t} \in \{2, 3\}$ for distinct $s, t \in S$; and recently by Charney [5] for "locally reducible" Artin groups (those in which every finite type special subgroup is a direct product of special subgroups of rank at most 2) which includes the triangle-free and extra-large cases, but none of the irreducible finite types of rank greater than 2. The arguments given here are self-contained and independent of these previous approaches to the problem.

We note that there is an obvious homomorphism f from the abstract group $H(\Gamma)$ defined by the presentation displayed in Theorem 1 to the group $A(\Gamma)$. Our task is to show that f is an isomorphism onto its image. We consider first the Artin groups of **small type**, those for which $m_{s,t} \in \{2,3\}$ if $s, t \in S$ are distinct. In Section 3 we describe, for each irreducible Artin group $A(\Gamma)$ of small type, a representation of $A(\Gamma)$ in the mapping class group of a connected surface $F(\Gamma)$ with boundary. This naturally induces an action of $A(\Gamma)$, and via the map f an action of $H(\Gamma)$, on the fundamental groupoid of $F(\Gamma)$ relative to a certain finite set of "basepoints". The result follows by showing that the action of $H(\Gamma)$ on this groupoid is faithful. This is done in Sections 4 and 5. The proof proceeds by induction on the length of so-called M-reduced expressions in the generators of $H(\Gamma)$, and relies also on the fact that the fundamental groupoid which we consider is isomorphic to a free groupoid on a graph. Notions concerning groupoids are discussed in Section 2.

We explain in Section 6 how one may construct a homomorphism from an arbitrary Artin group $A(\Gamma)$ to an Artin group $A(\widetilde{\Gamma})$ of small type such that the induced map $H(\Gamma) \to H(\widetilde{\Gamma})$ is injective. This reduces the general case of Theorem 1 to the small type case already discussed.

2   The free groupoid on a graph

A **groupoid** is a category in which every morphism is invertible and which is connected in the sense that there exists a morphism between any pair of objects. A group, for example, is a groupoid with exactly one object. The objects of a groupoid are called **vertices**. For every vertex x, the morphisms from x to itself form a group called the **vertex group**. An element is said to be **constant** if it is the identity of some vertex group, and **non-constant** otherwise.

To give another important example of a groupoid, let X be a path-connected topological space and let $\{P_i\}_{i \in I}$ be a discrete set of points in X. The **fundamental groupoid** of X based at $\{P_i\}_{i \in I}$ is the groupoid, denoted $\pi_1(X, \{P_i\}_{i \in I})$, whose vertex set is $\{P_i\}_{i \in I}$, and in which the morphisms from P_i to P_j are the homotopy classes of paths from P_i to P_j. Constant morphisms are represented by constant paths.

We now give a combinatorial construction which is motivated by considering fundamental groupoids of graphs (1-dimensional simplicial complexes).

An **oriented graph** G consists of the following:

(a) a set $V(G)$ of **vertices**;

(b) a set $E(G)$ of **edges**;

(c) a **source** function $\mathrm{src} : E(G) \to V(G)$ and a **target** function $\mathrm{trg} : E(G) \to V(G)$.

We define an abstract set $E(G)^{-1} = \{a^{-1} : a \in E(G)\}$ in bijection with $E(G)$ and write $\mathrm{src}(a^{-1}) = \mathrm{trg}(a)$ and $\mathrm{trg}(a^{-1}) = \mathrm{src}(a)$ for all $a \in E(G)$. A **path** in G is an expression

$$f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_k^{\varepsilon_k}$$

where $a_i \in E(G)$, $\varepsilon_i \in \{\pm 1\}$, and $\operatorname{trg}(a_i^{\varepsilon_i}) = \operatorname{src}(a_{i+1}^{\varepsilon_{i+1}})$. The vertex $\operatorname{src}(a_1^{\varepsilon_1})$ is called the **source** of g, written $\operatorname{src}(g)$, and $\operatorname{trg}(a_k^{\varepsilon_k})$ the **target** of g, written $\operatorname{trg}(g)$. The number k is the **length** of g. The vertices are considered as paths of length 0. The **inverse** of a path $f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_k^{\varepsilon_k}$ is the path $f^{-1} = a_k^{-\varepsilon_k} ... a_2^{-\varepsilon_2} a_1^{-\varepsilon_1}$. The **product** of two paths $f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_p^{\varepsilon_p}$ and $g = b_1^{\mu_1} b_2^{\mu_2} ... b_q^{\mu_q}$ such that $\operatorname{trg}(f) = \operatorname{src}(g)$ is the path $fg = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_p^{\varepsilon_p} b_1^{\mu_1} b_2^{\mu_2} ... b_q^{\mu_q}$. A path $f = a_1^{\varepsilon_1} a_2^{\varepsilon_2} ... a_k^{\varepsilon_k}$ is **reduced** if there is no $i \in \{1, 2, .., (k-1)\}$ such that $a_i = a_{i+1}$ and $\varepsilon_i = -\varepsilon_{i+1}$.

A **homotopy** on G is an equivalence relation, $\sim$, on the paths of G satisfying:

(a) $f \sim g \Rightarrow \operatorname{src}(f) = \operatorname{src}(g)$ and $\operatorname{trg}(f) = \operatorname{trg}(g)$;

(b) $f^{-1} f \sim \operatorname{src}(f)$ for all paths f;

(c) $f \sim g \Rightarrow f^{-1} \sim g^{-1}$;

(d) if $f \sim g$, $\operatorname{trg}(h_1) = \operatorname{src}(f) = \operatorname{src}(g)$ and $\operatorname{src}(h_2) = \operatorname{trg}(f) = \operatorname{trg}(g)$, then $h_1 f h_2 \sim h_1 g h_2$.

A homotopy, $\sim$, on G determines a groupoid $\pi(G, \sim)$ whose objects are the vertices of G and whose morphisms are the equivalence classes of paths. The **free groupoid** on G, denoted $\pi(G)$, is the groupoid determined by the smallest homotopy of G. Note that the morphisms of $\pi(G)$ are in bijection with the reduced paths. Since it is the usual convention to compose morphisms in a category on the left, one should associate the domain and codomain of a homotopy class of paths with the target and source of a representative, in that order. (This last point is purely a matter of convention, and of no significance in what follows). We will use, later, the fact that $\pi(G)$ is isomorphic to the fundamental groupoid $\pi_1(G, V(G))$ via the map defined by sending each edge e to the homotopy class $[\gamma_e]$ where γ_e denotes a path in G which travels along the edge e from $\operatorname{src}(e)$ to $\operatorname{trg}(e)$.

3   Representation in the Mapping Class Group of a surface

Let F be a compact, connected, oriented surface with boundary ($\partial F \neq \emptyset$). We denote by $\mathcal{H}(F)$ the group of homeomorphisms $h : F \to F$ which preserve the orientation and restrict to the identity on the boundary of F. The **mapping class group** of F is the group $\mathcal{M}(F)$ of isotopy classes in $\mathcal{H}(F)$.

If $\{P_i\}_{i \in I}$ is a finite collection of points on the boundary of F, then $\mathcal{M}(F)$ acts on the fundamental groupoid $\pi_1(F, \{P_i\}_{i \in I})$ by automorphisms.

An **essential circle** in F is an oriented embedding $a : S^1 \to F$ such that $a(S^1) \cap \partial F = \emptyset$. We shall use the description of the circle S^1 as $\mathbb{R}/2\pi\mathbb{Z}$. Take an (oriented) embedding $A : S^1 \times [0, 1] \to F$ of the annulus such that $A(\theta, \frac{1}{2}) = a(\theta)$ for all $\theta \in S^1$, and define a homeomorphism $\operatorname{Tw}_a \in \mathcal{H}(F)$ which restricts to the identity outside the interior of the image of A, and is otherwise given by

$$(\operatorname{Tw}_a \circ A)(\theta, x) = A(\theta + 2\pi x, x) \quad \text{for } (\theta, x) \in S^1 \times [0, 1].$$
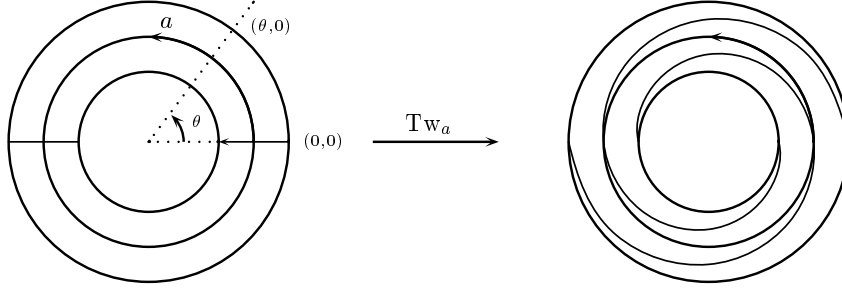
Figure 1: A Dehn twist

The **Dehn twist** along a is the element of $\mathcal{M}(F)$ represented by Tw_a (see Figure 1). The following result is easily checked and may be found, for instance, in [2].

Proposition 3. *Let $a_1, a_2 : S^1 \to F$ be two essential circles, and for $i = 1, 2$, let τ_i denote the Dehn twists along a_i. Then*

$$\begin{aligned}
\tau_1 \tau_2 &= \tau_2 \tau_1 & &\text{if } a_1 \cap a_2 = \emptyset \\
\tau_1 \tau_2 \tau_1 &= \tau_2 \tau_1 \tau_2 & &\text{if } |a_1 \cap a_2| = 1 \,.
\end{aligned}$$

We suppose now that Γ is a connected Coxeter graph of **small type**, which is to say that $m_{s,t} \in \{2, 3\}$ for distinct $s, t \in S$. We are first going to associate to Γ a surface $F(\Gamma)$.

Let $<$ denote a total order on S, which may be chosen arbitrarily. For each $s \in S$ we define the set ("star of s")

$$\mathrm{St}_s = \{t \in S : m_{s,t} = 3\} \cup \{s\} \,.$$

We write $\mathrm{St}_s = \{t_1, t_2, ..., t_k\}$ such that $t_1 < t_2 < ... < t_k$ and suppose that $s = t_j$. The difference $i - j$ will be called the **relative position** of t_i with respect to s and will be denoted $\mathrm{pos}(t_i; s)$. In particular, $\mathrm{pos}(s; s) = 0$.

Let $s \in S$. Put $k = \mathrm{card}(\mathrm{St}_s)$. We denote by An_s the annulus defined by

$$\mathrm{An}_s = \mathbb{R}/2k\mathbb{Z} \times [0, 1] \,,$$

For each $s \in S$, write P_s for the point $(0, 0)$ of An_s.

The surface $F(\Gamma)$ is defined by

$$F(\Gamma) = \Big(\coprod_{s \in S} An_s \Big) \Big/_{\approx} \,,$$

where $\approx$ is the relation defined as follows. Let $s, t \in S$ such that $m_{s,t} = 3$ and $s < t$. Put $p = \mathrm{pos}(t; s) > 0$ and $q = \mathrm{pos}(s; t) < 0$. For each $(x, y) \in [0, 1] \times [0, 1]$, the relation $\approx$ identifies the point $(2p + x, y)$ of An_s with the point $(2q + 1 - y, x)$ of An_t (see Figure 2).
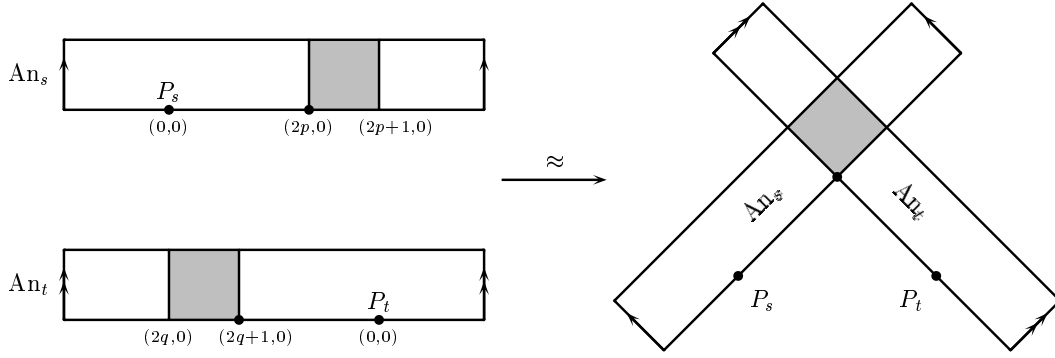
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Figure 2: Identification of annuli

We identify each annulus An_s and the point P_s with their images in $F(\Gamma)$ respectively. We also define the set $\mathcal{P} = \{P_s\}_{s \in S}$ in the boundary of $F(\Gamma)$ to be used as the set of basepoints for a fundamental groupoid.

We now define a group homomorphism $g : A(\Gamma) \to \mathcal{M}(F(\Gamma))$. Let $s \in S$, and put $k = \text{card}(\text{St}_s)$. We denote by $a_s : S^1 \to F(\Gamma)$ the essential circle of $F(\Gamma)$ such that $a_s(\theta)$ is the point $(\frac{k\theta}{\pi}, \frac{1}{2})$ of An_s (see Figure 3).
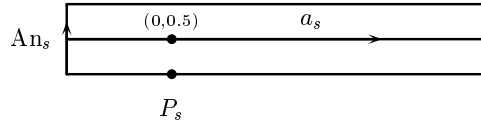

Figure 3: The essential circle a_s in An_s

We let τ_s denote the Dehn twist along a_s. One has $a_s \cap a_t = \emptyset$ if $m_{s,t} = 2$, and $|a_s \cap a_t| = 1$ if $m_{s,t} = 3$. Therefore, by Proposition 3, we have the following.

Proposition 4. *Let Γ be a connected Coxeter graph of small type. There exists a well-defined group homomorphism $g : A(\Gamma) \to \mathcal{M}(F(\Gamma))$ which sends σ_s to τ_s for each $s \in S$. In particular, the group $A(\Gamma)$ acts (via the homomorphism g and the natural action of the mapping class group) by automorphisms of the fundamental groupoid $\pi_1(F(\Gamma), \mathcal{P})$.*

We finish this Section with a description of the fundamental groupoid $\pi_1(F(\Gamma), \mathcal{P})$ as the free groupoid of a certain graph. One associates to Γ an oriented graph $G(\Gamma)$ as follows. First let

$$\mathcal{B} = \{(s,t) \in S \times S : m_{s,t} = 3 \text{ and } s < t\}.$$

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Then put

$$V(G(\Gamma)) = S\,, \text{ and}$$
$$E(G(\Gamma)) = \{e_s : s \in S\} \cup \{f_{s,t} : (s,t) \in \mathcal{B}\}$$

where $\mathrm{src}(e_s) = \mathrm{trg}(e_s) = s$, for all $s \in S$, and $\mathrm{src}(f_{s,t}) = s$ and $\mathrm{trg}(f_{s,t}) = t$, for all $(s,t) \in \mathcal{B}$.

We turn now to the fundamental groupoid. Fix $s \in S$ and let $k = \mathrm{card}(\mathrm{St}_s)$. Then we define $\alpha_s : [0,1] \to F(\Gamma)$ to be the loop at P_s such that $\alpha_s(x)$ is the point $(2kx, 0)$ of An_s (see Figure 4).
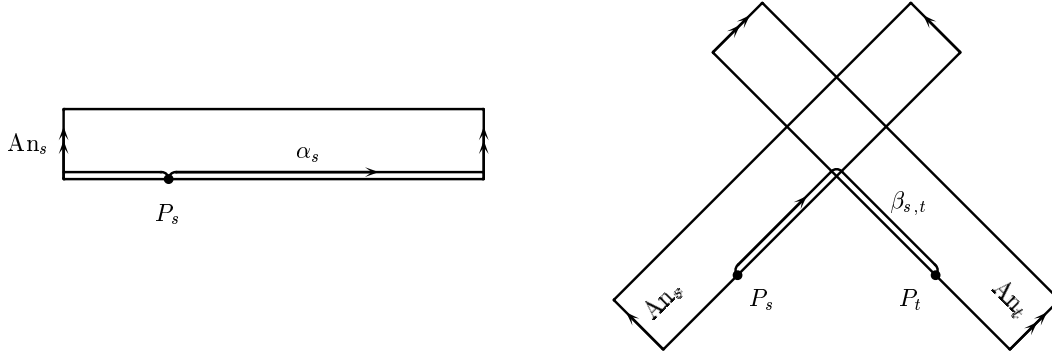

Figure 4: The loop α_s and the path $\beta_{s,t}$

Fix $(s,t) \in \mathcal{B}$ and let $p = \mathrm{pos}(t;s) > 0$ and $q = \mathrm{pos}(s;t) < 0$. We define $\beta_{s,\bullet} : [0,1] \to F(\Gamma)$ to be the path such that $\beta_{s,\bullet}(x)$ is the point $(2px, 0)$ of An_s and $\beta_{\bullet,t} : [0,1] \to F(\Gamma)$ the path such that $\beta_{\bullet,t}(x)$ is the point $((2q+1)(1-x), 0)$ of An_t. Then $\beta_{s,t} : [0,1] \to F(\Gamma)$ is the path from P_s to P_t defined by $\beta_{s,t} = \beta_{s,\bullet}\beta_{\bullet,t}$ (see Figure 4).

We now have the following proposition.

Proposition 5. *There is an isomorphism $\pi(G(\Gamma)) \to \pi_1(F(\Gamma), \mathcal{P})$ which sends e_s to $[\alpha_s]$ for all $s \in S$, and $f_{s,t}$ to $[\beta_{s,t}]$ for all $(s,t) \in \mathcal{B}$.*

Proof. Recall that, for an oriented graph G, the free groupoid $\pi(G)$ is canonically isomorphic to the fundamental groupoid $\pi_1(G, V(G))$. The Proposition now follows from the observation that the map $G(\Gamma) \to F(\Gamma)$ which sends the edge e_s to the loop α_s, for each $s \in S$, and the edge $f_{s,t}$ to the path $\beta_{s,t}$, for each $(s,t) \in \mathcal{B}$, is a homotopy equivalence relative to the vertex set S and its image $\{P_s\}_{s \in S}$. ●

4   Representation by automorphisms of a free groupoid

Suppose still that Γ is a connected Coxeter graph of small type i.e: $m_{s,t} \in \{2,3\}$ for distinct $s,t \in S$. We turn now to study in detail the action of the Artin group $A(\Gamma)$ on

the free groupoid $\pi_1(F(\Gamma), \mathcal{P})$. For convenience, we shall use α_s and $\beta_{s,t}$ to denote the groupoid generators represented by the paths of the same name defined in the previous section. We also use τ_s to denote both the Dehn twist along the midline a_s of An_s and the groupoid automorphism which this Dehn twist induces.

Let $B = \langle \beta_{s,t} : (s,t) \in \mathcal{B} \rangle$ denote the subgroupoid of $\pi_1(F(\Gamma), \mathcal{P})$ generated by the set $\{ \beta_{s,t} : (s,t) \in \mathcal{B} \}$. This subgroupoid can also be described as follows. Let $G_0(\Gamma)$ denote the subgraph of $G(\Gamma)$ having the same vertex set, but including only the edges $f_{s,t}$ for $(s,t) \in \mathcal{B}$. Then one has an isomorphism $\pi(G_0(\Gamma)) \to B$ sending $f_{s,t}$ to $\beta_{s,t}$ for all $(s,t) \in \mathcal{B}$.

Observe that any element $w \in \pi_1(F(\Gamma), \mathcal{P})$ has a unique expression of the form

$$ w = \mu_0 \alpha_{s_1}^{k_1} \mu_1 ... \alpha_{s_\ell}^{k_\ell} \mu_\ell $$

where $\mu_i \in B$, μ_i is non-constant if $i \notin \{0, \ell\}$, $s_i \in S$, and $k_i \in \mathbb{Z} \setminus \{0\}$. We call this expression the **reduced form** of w.

Let $s \in S$. Given the reduced form as above, we shall say that w **has a square in** α_s if there exists $i \in \{1, .., \ell\}$ such that $s_i = s$ and $|k_i| \geq 2$. Otherwise, we say that w **is without squares in** α_s.

Now let $\mu \in B$, $t \in S$, and $m \in \mathbb{Z} \setminus \{0\}$. we say that w **is of type** (μ, α_t^m) if w has reduced form

$$ w = \mu_0 \alpha_t^{k_1 m} \mu_1 ... \alpha_t^{k_\ell m} \mu_\ell $$

where $\mu_i \in B$, μ_i is non-constant if $i \notin \{0, \ell\}$, $k_i \in \mathbb{Z} \setminus \{0\}$, and $\mu = \mu_0 \mu_1 ... \mu_\ell$. Note that, for fixed $t \in S$ and $m \in \mathbb{Z} \setminus \{0\}$, the set $B(\alpha_t^m)$ of all elements of type (μ, α_t^m) (for some $\mu \in B$) forms a subgroupoid of $\pi_1(F(\Gamma), \mathcal{P})$ which admits a homomorphism onto B sending elements of type (μ, α_t^m) to μ.

Recall that, for $t \in S$, the generator σ_t of $A(\Gamma)$ acts on $\pi_1(F(\Gamma), \mathcal{P})$ via the Dehn twist τ_t along the midline a_t of An_t. As it will be useful later, in Section 5, we now consider the action on the groupoid of an arbitrary power of τ_t (rather than simply τ_t itself). We have the following:

Proposition 6. *Let $t \in S$ and $m \in \mathbb{Z} \setminus \{0\}$.*

(i) Let $\mu \in B$. Then $\tau_t^m(\mu)$ is of type (μ, α_t^m).

(ii) Let $s \in S$. If $s = t$ or if $m_{s,t} = 2$, then

$$ \tau_t^m(\alpha_s) = \alpha_s . $$

If $m_{s,t} = 3$, then

$$ \tau_t^m(\alpha_s) = \begin{cases} u\alpha_s & \text{if } s < t \\ \alpha_s v & \text{if } t < s \, , \end{cases} $$

where u, respectively v, is a non-constant element of $\pi_1(F(\Gamma), \mathcal{P})$ of type $(1, \alpha_t^m)$. In particular, if $|m| \geq 2$, then $\tau_t^m(\alpha_s)$ has a square in α_t.
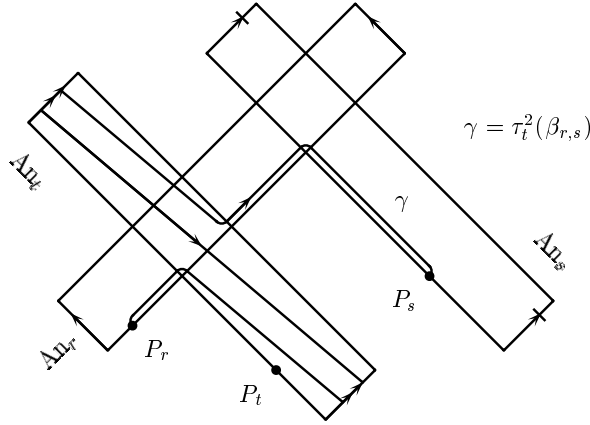
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Figure 5: The path $\tau_t^2(\beta_{r,s})$ when $t \in \mathrm{St}_r$ but $t \notin \mathrm{St}_s$.

Proof. (i) It suffices to consider the case where $\mu = \beta_{r,s}$ for some $(r,s) \in \mathcal{B}$. When $a_t \cap \beta_{r,s} = \emptyset$, we have $\tau_t^m(\beta_{r,s}) = \beta_{r,s}$ as required. On the other hand, we have $a_t \cap \beta_{r,s} \neq \emptyset$ if and only if $r < t < s$ and either $t \in \mathrm{St}_r$ or $t \in \mathrm{St}_s$. (Recall that $\mathrm{St}_s = \{t \in S : m_{s,t} = 3\} \cup \{s\}$, for $s \in S$.)

Suppose that $t \in \mathrm{St}_r$, but $t \notin \mathrm{St}_s$. Then

$$\tau_t^m(\beta_{r,s}) = \beta_{r,t}\alpha_t^m\beta_{r,t}^{-1}\beta_{r,s} \qquad \text{(see Figure 5)}.$$

Suppose that $t \in \mathrm{St}_s$, but $t \notin \mathrm{St}_r$. Then

$$\tau_t^m(\beta_{r,s}) = \beta_{r,s}\beta_{t,s}^{-1}\alpha_t^{-m}\beta_{t,s} \qquad \text{(see Figure 6)}.$$

Finally, suppose that $t \in \mathrm{St}_r \cap \mathrm{St}_s$. Then

$$\tau_t^m(\beta_{r,s}) = \beta_{r,t}\alpha_t^m\beta_{r,t}^{-1}\beta_{r,s}\beta_{t,s}^{-1}\alpha_t^{-m}\beta_{t,s} \qquad \text{(see Figure 7)}.$$

(ii) We have $\alpha_s \cap a_t \neq \emptyset$ if and only if $m_{s,t} = 3$. In particular, if $s = t$ or if $m_{s,t} = 2$, we have $\tau_t^m(\alpha_s) = \alpha_s$.

Suppose that $m_{s,t} = 3$. If $s < t$, then

$$\tau_t^m(\alpha_s) = \beta_{s,t}\alpha_t^m\beta_{s,t}^{-1}\alpha_s \qquad \text{(see Figure 8)}.$$

If $s > t$, then

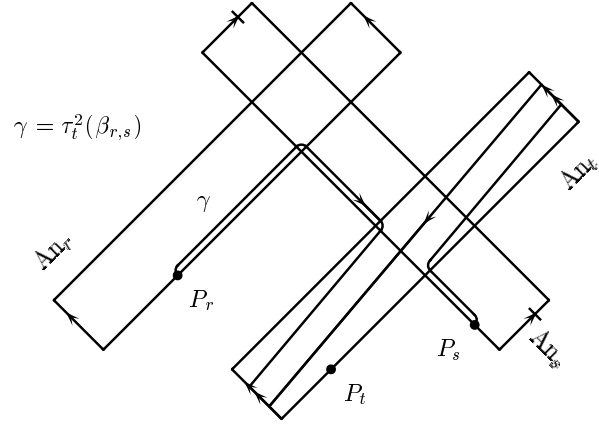$$\tau_t^m(\alpha_s) = \alpha_s\beta_{t,s}^{-1}\alpha_t^{-m}\beta_{t,s} \qquad \text{(see Figure 9)}.$$

$\bullet$

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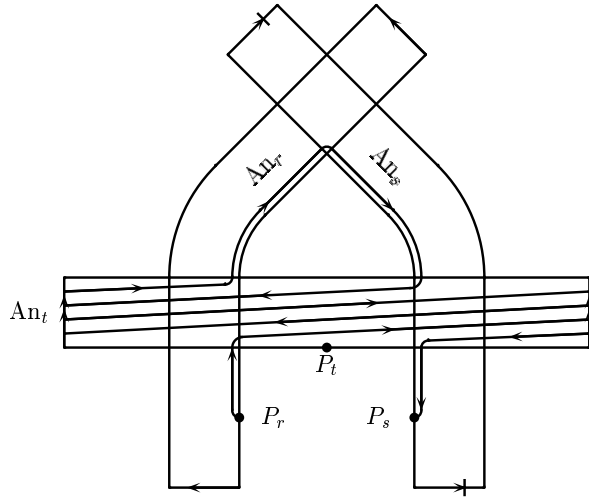Figure 6: The path $\tau_t^2(\beta_{r,s})$ when $t \notin \mathrm{St}_r$ but $t \in \mathrm{St}_s$.

Figure 7: The path $\tau_t^2(\beta_{r,s})$ when $t \in \mathrm{St}_s \cap \mathrm{St}_r$.
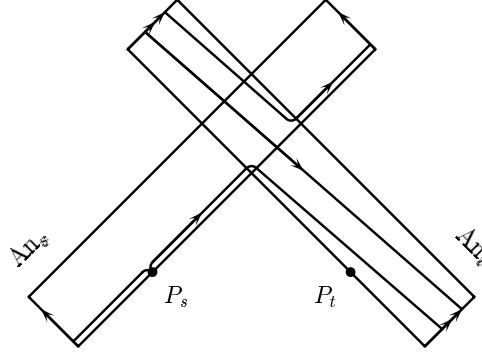
Figure 8: The loop $\tau_t^2(\alpha_s)$ when $t \in \mathrm{St}_s$ and $s < t$.

Proposition 7. *Let $x \in \pi_1(F(\Gamma), \mathcal{P})$, and let $s, t \in S$ and $m \in \mathbb{Z}$ such that $|m| \geq 2$. Suppose that x is without squares in α_t and that $\tau_t^m(x)$ has a square in α_s. Then one of the following two conditions hold:*

(a) $s = t$;

(b) $m_{s,t} = 2$ and x has a square in α_s.

Proof. Let $x = \mu_0 \alpha_{s_1}^{k_1} \mu_1 ... \alpha_{s_\ell}^{k_\ell} \mu_\ell$ in reduced form. Since x is without squares in α_t, we have $k_i = \pm 1$ if $s_i = t$. By Proposition 6, the reduced form of $\tau_t^m(x)$ may be written

$$\tau_t^m(x) = x_0 v_1 x_1 ... v_\ell x_\ell$$

where:

- x_i is a reduced form of type (μ_i, α_t^m) for all $i = 1, ..., \ell$;

- if $s_i = t$, then $v_i = \alpha_{s_i}^{p_i}$ where $p_i \equiv \pm 1 \pmod{m}$;

- if $m_{s_i,t} = 2$, then $v_i = \alpha_{s_i}^{k_i}$;

- if $m_{s_i,t} = 3$, then $v_i = \alpha_{s_i}^{\varepsilon_i}(v_i'\alpha_{s_i}^{\varepsilon_i})^{|k_i|-1}$ where $\varepsilon_i \in \{\pm 1\}$ and v_i' is a non-constant reduced form of type $(1, \alpha_t^m)$.

Note that in the last case, when $m_{s_i,t} = 3$, we actually have $\tau_t^m(\alpha_{s_i}^{k_i}) = (v_i'\alpha_{s_i}^{\varepsilon_i})^{|k_i|}$ or $(\alpha_{s_i}^{\varepsilon_i}v_i')^{|k_i|}$ depending on whether $s_i < t$ or not and on whether k_i is positive or negative. In each case, the extra v_i' is absorbed into the adjacent form x_{i-1} or x_i respectively.

It now follows that, if $\tau_t^m(x)$ has a square in α_s, then either $s = t$ or there exists $i \in \{1, ..., \ell\}$ such that $m_{s_i,t} = 2$ and $|k_i| \geq 2$. ●
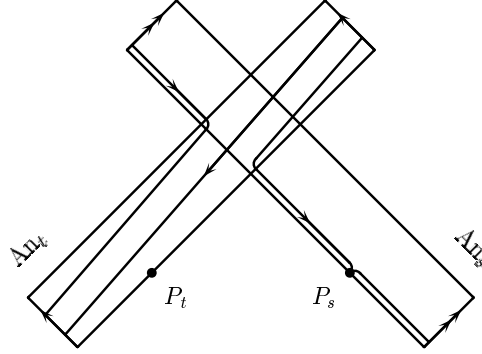
Figure 9: The loop $\tau_t^2(\alpha_s)$ when $t \in \mathrm{St}_s$ and $s > t$.

5 The small type case

For the moment, suppose that Γ is an arbitrary Coxeter graph. Take an abstract set $\mathcal{T} = \{T_s : s \in S\}$ in bijection with S, and consider the group $H(\Gamma)$ defined by the presentation

$$H(\Gamma) = \langle \mathcal{T} \mid T_s T_t = T_t T_s \text{ whenever } m_{s,t} = 2 \rangle .$$

Note that this presentation depends only on the unlabelled Coxeter graph, and that $H(\Gamma) = A(\Gamma_\infty)$ where Γ_∞ is the Coxeter graph obtained from Γ by relabelling every edge with ∞. The class of groups having such a presentation, known as "graph groups" or "right-angled Artin groups", have been widely studied. They are examples of so-called "graph products" of groups (namely the graph products of infinite cyclic groups) which were studied in the thesis of E.R. Green [10] (see also [11]).

Suppose that, for each $s \in S$, we are given an integer $m_s \geq 2$. We then have a homomorphism $f : H(\Gamma) \to A(\Gamma)$ which sends T_s to $\sigma_s^{m_s}$ for all $s \in S$. It is our objective in this paper (in proving Theorem 1) to show that this homomorphism f is always injective. In this section, we shall treat the small type case.

We begin with some general notions concerning the presentation given above for the group $H(\Gamma)$. An **expression** in $\mathcal{T}$ is a sequence

$$W = (T_{s_\ell}^{p_\ell}, ..., T_{s_2}^{p_2}, T_{s_1}^{p_1})$$

where $s_i \in S$, $p_i \in \mathbb{Z} \setminus \{0\}$. The number ℓ is called the **length** of the expression W, written $\ell = \ell(W)$. We write $[W]$ for the corresponding element $T_{s_\ell}^{p_\ell} ... T_{s_2}^{p_2} T_{s_1}^{p_1} \in H(\Gamma)$. Given $w \in H(\Gamma)$, we say that W is **an expression for** w if $w = [W]$. The following terminology is motivated by that used by Brown in [3].

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Consider the expression W as written above. Suppose that there exists $i \in \{1, ..., \ell-1\}$ such that $s_i = s_{i+1}$. Put

$$W' = \begin{cases} (..., T_{s_{i+2}}^{p_{i+2}}, T_{s_i}^{p_{i+1}+p_i}, T_{s_{i-1}}^{p_{i-1}}, ...) & \text{if } p_i + p_{i+1} \neq 0, \\ (..., T_{s_{i+2}}^{p_{i+2}}, T_{s_{i-1}}^{p_{i-1}}, ...) & \text{if } p_i + p_{i+1} = 0. \end{cases}$$

We say that W' is obtained from W via an **elementary M-operation of type I**. This operation shortens the length of an expression by 1 or 2.

Suppose that there exists $i \in \{1, ..., \ell-1\}$ such that $m_{s_i, s_{i+1}} = 2$. Put

$$W'' = (..., T_{s_{i+2}}^{p_{i+2}}, T_{s_i}^{p_i}, T_{s_{i+1}}^{p_{i+1}}, T_{s_{i-1}}^{p_{i-1}}, ...)$$

We say that W'' is obtained from W via an **elementary M-operation of type II**. This operation leaves the length of an expression unchanged.

We shall say that W is **M-reduced** if the length of W can not be reduced by applying a sequence of elementary M-operations. Clearly, every element of $H(\Gamma)$ has an M-reduced expression. If two expressions W and W' are related by a sequence of elementary M-operations of type II, then we say that they are **II-equivalent** and write $W \sim_{\mathrm{II}} W'$.

Let $s \in S$. An M-reduced expression is said to **end in** s if it is II-equivalent to an expression $(T_{s_\ell}^{p_\ell}, ..., T_{s_1}^{p_1})$ in which $s_\ell = s$.

Lemma 8. *Suppose that $W = (T_{s_\ell}^{p_\ell}, ..., T_{s_2}^{p_2}, T_{s_1}^{p_1})$ is an expression, and fix an integer r such that $2 \leq r \leq \ell$. Then the following are equivalent:*

(i) W is M-reduced;

(ii) for all $i < j$ such that $s_i = s_j$, there exists k such that $i < k < j$ and $m_{s_i, s_k} \neq 2$;

(iii) the expressions $U = (T_{s_r}^{-p_r}, ..., T_{s_\ell}^{-p_\ell})$ and $V = (T_{s_{r-1}}^{p_{r-1}}, ..., T_{s_1}^{p_1})$ are both M-reduced, and there is no $s \in S$ such that both U and V end in s.

Proof. That both (ii) and (iii) are consequences of (i) is obvious. We note that the truth of condition (ii) is invariant under elementary M-operations of type II, while an elementary M-operation of type I can never be performed on an expression satisfying (ii). Thus (ii)$\Rightarrow$(i). We now show that (iii)$\Rightarrow$(ii). Suppose that (iii) holds, but that there exist $i < j$ such that $s_i = s_j$ and $m_{s_i, s_k} = 2$ for all $i < k < j$. If $i < j < r$, or $r \leq i < j$, then we contradict that both V and U are M-reduced, while if $i < r \leq j$ then it is clear that both U and V end in s_i. $\qquad\qquad\bullet$

We return now to the case where Γ is a connected Coxeter graph of small type. Recall the homomorphism $g : A(\Gamma) \to \mathcal{M}(F(\Gamma))$ of Proposition 4. We now consider the group $H(\Gamma)$ to act on $\pi_1(F(\Gamma), \mathcal{P})$ via the homomorphism $g \circ f : H(\Gamma) \to A(\Gamma) \to \mathcal{M}(F(\Gamma))$.

Proposition 9. *Let W be an M-reduced expression for $w \in H(\Gamma)$. Let $x \in \pi_1(F(\Gamma), \mathcal{P})$, and $t \in S$. Suppose that x is without squares in α_s for all $s \in S$ and that $w(x)$ has a square in α_t. Then W ends in t.*

Proof. We proceed by induction on the length of W. Let W be written $(T_{s_\ell}^{p_\ell}, ..., T_{s_1}^{p_1})$ and write $W' = (T_{s_{\ell-1}}^{p_{\ell-1}}, ..., T_{s_1}^{p_1})$. If $[W'](x)$ has a square in α_{s_ℓ} then, by the induction hypothesis, W' ends in s_ℓ. But then W is II-equivalent to an expression $(T_{s_\ell}^{p_\ell}, T_{t_{\ell-1}}^{q_{\ell-1}}, ..., T_{t_1}^{q_1})$, where $t_{\ell-1} = s_\ell$. But this expression admits an elementary operation of type I, contradicting the assumption that W is M-reduced. Thus, $[W'](x)$ is without squares in α_{s_ℓ}.

Note that $w(x) = \tau_{s_\ell}^m([W'](x))$ where $m = p_\ell m_{s_\ell}$ and $|m| \geq 2$. We are given that $w(x)$ has a square in α_t. Therefore, by Proposition 7, one either has that $s_\ell = t$ or that $m_{s_\ell, t} = 2$ and $[W'](x)$ has a square in α_t. If $s_\ell = t$, then W obviously ends in t. We may therefore suppose the latter, in which case, by induction, W' ends in t. In other words, W' is II-equivalent to an expression $(T_{t_{\ell-1}}^{q_{\ell-1}}, ..., T_{t_1}^{q_1})$ with $t_{\ell-1} = t$. Then, using the fact that $m_{s_\ell, t} = 2$,

$$W \sim_{\mathrm{II}} (T_{s_\ell}^{p_\ell}, T_t^{q_{\ell-1}}, T_{t_{\ell-2}}^{q_{\ell-2}}, ..., T_{t_1}^{q_1}) \sim_{\mathrm{II}} (T_t^{q_{\ell-1}}, T_{s_\ell}^{p_\ell}, T_{t_{\ell-2}}^{q_{\ell-2}}, ..., T_{t_1}^{q_1}).$$

Hence W ends in t. ●

Proposition 10. *If there exists a nontrivial M-reduced expression for $w \in H(\Gamma)$, then $f(w)$ is nontrivial. In particular, the homomorphism $f : H(\Gamma) \to A(\Gamma)$ is injective.*

Proof. Note that the result is obvious in the case that Γ is a trivial graph (i.e: Γ consists of just one vertex, and both $H(\Gamma)$ and $A(\Gamma)$ are infinite cyclic). Suppose then that Γ is nontrivial, and let $W = (T_{s_\ell}^{p_\ell}, ..., T_{s_1}^{p_1})$ be a nontrivial M-reduced expression for $w \in H(\Gamma)$. Using the fact that Γ is connected and nontrivial, choose $t \in S$ such that $m_{s_\ell, t} = 3$. We shall show that $w(\alpha_t) \neq \alpha_t$. This clearly implies that $f(w) \neq 1$.

Suppose that $w(\alpha_t) = \alpha_t$. Putting $W' = (T_{s_{\ell-1}}^{p_{\ell-1}}, ..., T_{s_1}^{p_1})$, we have

$$[W'](\alpha_t) = T_{s_\ell}^{-p_\ell}(\alpha_t) = \tau_{s_\ell}^m(\alpha_t)$$

where $m = -p_\ell m_{s_\ell}$, and $|m| \geq 2$. By Proposition 6, $\tau_{s_\ell}^m(\alpha_t)$ has a square in α_{s_ℓ}. It follows, by Proposition 9, that W' ends in s_ℓ. But this contradicts W being M-reduced. Therefore $w(\alpha_t) \neq \alpha_t$. ●

Corollary 11. *Theorem 1 and the Tits Conjecture hold for irreducible Artin groups of small type.*

It has been known for some time that the finite process of obtaining an M-reduced expression from any given expression (via elementary M-operations) provides a solution to the word problem of the graph group $H(\Gamma)$. Namely:

> If W is an M-reduced expression representing the identity then W is trivial. (1)

We refer to [10], and more recently [12], both of which treat the more general case of graph products. We point out that, none of our arguments rely upon this solution to the word problem. On the contrary, statement (1) is actually a *consequence* of Proposition 10 (for graph groups only). We note also that, by using Lemma 8(iii) and induction on the lengths of expressions, one can show the strengthened version of (1), as stated in [11], that any two M-reduced expressions for the same element are II-equivalent.

6 The general case

We introduce the Coxeter graph A_n (for $n > 0$) of Figure 10, and number the vertices of A_n as shown. The Artin group $A(A_n)$ is better known as the braid group on $(n+1)$ strings.
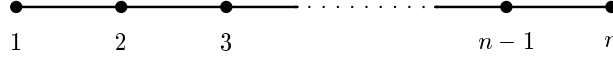

Figure 10: The Coxeter graph A_n.

We extend the notation $\mathrm{prod}(-,-;m)$ defined in the introduction to apply to arbitrary expressions x, y. Thus $\mathrm{prod}(x, y; m) = (xy)^{\frac{m}{2}}$ if m is even, and $x(yx)^{\frac{m-1}{2}}$ if m is odd. The following result can be found in the paper of Brieskorn and Saito [4], Lemma 5.8.

Proposition 12. *For $n \in \mathbb{N} \setminus \{0\}$, let $\{n, n-1\} = \{k, \ell\}$ where k is odd and ℓ is even, and let $P = \sigma_1\sigma_3...\sigma_k$ and $Q = \sigma_2\sigma_4...\sigma_\ell$ (or 1 if $n = 1$) denote elements of $A(A_n)$. Then*

$$\mathrm{prod}(P, Q; n+1) = \mathrm{prod}(Q, P; n+1).$$

The element $\mathrm{prod}(P, Q; n+1)$ of $A(A_n)$ is the so-called 'fundamental element' introduced into the study of braid groups by Garside [9].

Let m be an integer, $m \geq 3$. Let $\Gamma(m)$ denote the Coxeter graph with $2(m-1)$ vertices illustrated in Figure 11. This is a bi-partite graph with vertex set $I \sqcup J$, where $|I| = |J| = m - 1$ as shown in the figure. As a Coxeter graph, $\Gamma(m)$ is the disjoint union of two copies of A_{m-1}. Thus $A(\Gamma(m)) \cong A(A_{m-1}) \times A(A_{m-1})$.
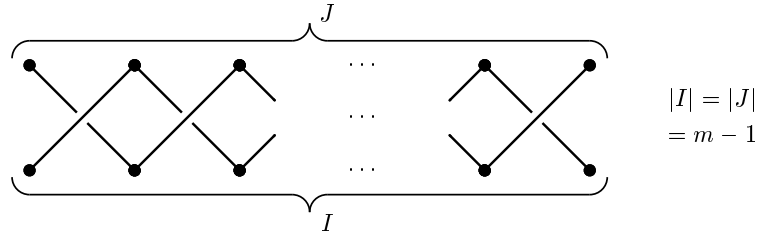

Figure 11: The Coxeter graph $\Gamma(m)$.

Let α, β be the elements of $A(\Gamma(m))$ defined by

$$\alpha = \prod_{i \in I} \sigma_i \qquad \text{and} \qquad \beta = \prod_{j \in J} \sigma_j$$

Then, by Proposition 12, we have

$$\mathrm{prod}(\alpha, \beta; m) = \mathrm{prod}(\beta, \alpha; m).$$
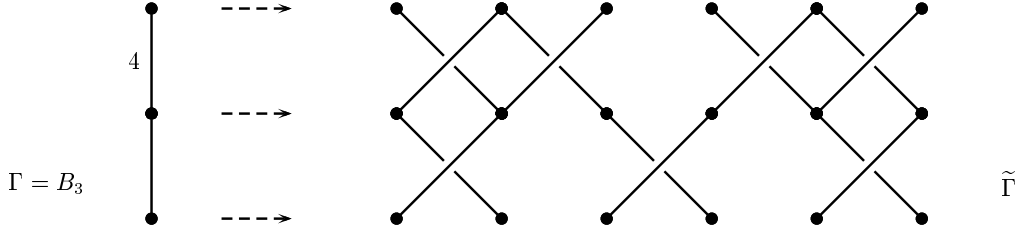
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Figure 12: A graph $\widetilde{\Gamma}$ for $\Gamma = B_3$.

Let $k \in \mathbb{N}$. We let $k\Gamma(m)$ denote the disjoint union of k copies of $\Gamma(m)$, which is a bi-partite graph with vertex set $kI \sqcup kJ$, where kI denotes the k disjoint copies of I, and kJ the k disjoint copies of J. Now let α, β be the elements of $A(k\Gamma(m))$ defined by

$$\alpha = \prod_{i \in kI} \sigma_i \qquad \text{and} \qquad \beta = \prod_{j \in kJ} \sigma_j$$

We still have

$$\mathrm{prod}(\alpha, \beta; m) = \mathrm{prod}(\beta, \alpha; m).$$

We suppose now that Γ is a connected Coxeter graph in which $m_{s,t} \neq \infty$ for all $s, t \in S$. Let N denote the least common multiple of the set $\{m_{s,t} - 1 : s \neq t \in S\}$ of natural numbers. For each $s \in S$, take an abstract set $I(s)$ with N elements. We define a Coxeter graph $\widetilde{\Gamma}$ of small type as follows:

- the vertex set of $\widetilde{\Gamma}$ is the disjoint union of the sets $I(s)$ for $s \in S$;

- if $m_{s,t} = 2$, then there are no edges between the vertices of $I(s)$ and $I(t)$;

- if $m_{s,t} \geq 3$, then the full subgraph of $\widetilde{\Gamma}$ spanned by $I(s) \sqcup I(t)$ is isomorphic to $(\frac{N}{m_{s,t}-1})\Gamma(m_{s,t})$ by a graph isomorphism taking $I(s)$ to $(\frac{N}{m_{s,t}-1})I$ and $I(t)$ to $(\frac{N}{m_{s,t}-1})J$.

Such a graph always exists, but is not unique in general. An example is given in Figure 12.

By the preceding remarks, we now have:

Proposition 13. *There exists a homomorphism* $\phi : A(\Gamma) \to A(\widetilde{\Gamma})$ *which sends* σ_s *to* $\prod_{i \in I(s)} \sigma_i$ *for each* $s \in S$.

Remark. The homomorphism of Proposition 13 is an example of a homomorphism obtained by a "folding" of $\widetilde{\Gamma}$ onto Γ as described in [7]. It was shown in [7] that the corresponding homomorphism ϕ^+ between Artin monoids is injective, and it seems quite possible (though this remains a conjecture) that ϕ is also injective.

Recall that one is given an integer $m_s \geq 2$ for each $s \in S$ and that $f : H(\Gamma) \to A(\Gamma)$ denotes the homomorphism which sends T_s to $\sigma_s^{m_s}$ for each $s \in S$.

Proposition 14. *Let Γ be a connected Coxeter graph with $m_{s,t} \neq \infty$ for all $s,t \in S$. Then the homomorphism $f : H(\Gamma) \to A(\Gamma)$ is injective.*

Proof. Consider the homomorphism $\widetilde{f} : H(\widetilde{\Gamma}) \to A(\widetilde{\Gamma})$ which sends T_i to $\sigma_i^{m_s}$ for all $s \in S$ and all $i \in I(s)$, and the homomorphism $\psi : H(\Gamma) \to H(\widetilde{\Gamma})$ which sends T_s to $\prod_{i \in I(s)} T_i$. Then we clearly have $\widetilde{f} \circ \psi = \phi \circ f$. Since $\widetilde{f}$ is injective by Proposition 10, it suffices to show that ψ is injective.

Let $W = (T_{s_\ell}^{p_\ell}, ..., T_{s_2}^{p_2}, T_{s_1}^{p_1})$ be an M-reduced expression for a nontrivial element $w \in H(\Gamma)$. Label the vertices of Γ so that $I(s) = \{s(1), s(2), ..., s(N)\}$, for each $s \in S$. Then the expression

$$U = (T_{s_\ell(N)}^{p_\ell}, .., T_{s_\ell(1)}^{p_\ell}, ..., T_{s_2(N)}^{p_2}, .., T_{s_2(1)}^{p_2}, T_{s_1(N)}^{p_1}, .., T_{s_1(1)}^{p_1})$$

is an expression for $\psi(w)$, and is M-reduced by condition (ii) of Lemma 8. Clearly U is nontrivial (since W is), and so, by Proposition 10, $\psi(w) \neq 1$. $\bullet$

Proposition 15. *Let Γ be an arbitrary Coxeter graph. Then the homomorphism $f : H(\Gamma) \to A(\Gamma)$ is injective.*

Proof. First suppose that Γ is connected. We define $\widehat{\Gamma}$ to be the Coxeter graph associated to the Coxeter matrix $\widehat{M} = (\widehat{m}_{s,t})_{s,t \in S}$ where

$$\widehat{m}_{s,t} = \begin{cases} 3 & \text{if } m_{s,t} = \infty \\ m_{s,t} & \text{otherwise.} \end{cases}$$

We have a homomorphism $\phi : A(\Gamma) \to A(\widehat{\Gamma})$ sending σ_s to σ_s for each $s \in S$, and a homomorphism $\widehat{f} : H(\Gamma) = H(\widehat{\Gamma}) \to A(\widehat{\Gamma})$ sending T_s to $\sigma_s^{m_s}$ for each $s \in S$. Since $\widehat{f} = \phi \circ f$ and $\widehat{f}$ is injective, by Proposition 14, it follows that f is injective.

Finally, suppose that Γ is a disjoint union of connected subgraphs Γ_i for $i = 1, .., k$. Then $A(\Gamma)$ and $H(\Gamma)$ are direct products

$$A(\Gamma) = \prod_{i=1}^{k} A(\Gamma_i), \quad \text{and} \quad H(\Gamma) = \prod_{i=1}^{k} H(\Gamma_i),$$

and the homomorphism f is a product $f_1 \times f_2 \times \cdots \times f_k$ of injective homomorphisms $f_i : H(\Gamma_i) \to A(\Gamma_i)$. Thus f is injective. $\bullet$

This completes the proof of Theorem 1 and the Tits conjecture for an arbitrary Artin group.

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